

# A NOTE ON THE AUTOMORPHISM GROUP OF A COMPACT COMPLEX MANIFOLD

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ABSTRACT. In this note, we give explicit examples of compact complex 3-folds which admit automorphisms that are isotopic to the identity through  $C^\infty$ -diffeomorphisms but not through biholomorphisms. These automorphisms play an important role in the construction of the Teichmüller stack of higher dimensional manifolds.

## 1. INTRODUCTION.

Let  $X$  be a compact complex manifold and  $M$  the underlying  $C^\infty$  manifold. The automorphism group  $\text{Aut}(X)$  of  $X$  is a complex Lie group whose Lie algebra is the Lie algebra of holomorphic vector fields [1]. We denote by  $\text{Aut}^0(X)$  the connected component of the identity. Its elements are thus automorphisms  $f$  such that there exists a  $C^\infty$ -isotopy

$$(1.1) \quad t \in [0, 1] \mapsto f_t \in \text{Aut}(X) \quad \text{with } f_0 = \text{Id} \text{ and } f_1 = f.$$

Note that  $\text{Aut}^0(X)$  has at most a countable number of connected components so the quotient  $\text{Aut}(X)/\text{Aut}^0(X)$  is discrete.

Let  $\text{Diff}(M)$  be the Fréchet Lie group of  $C^\infty$ -diffeomorphisms of  $M$ . It is tangent at the identity to the Lie algebra of  $C^\infty$  vector fields. Let  $\text{Diff}^0(M)$  be the connected component of the identity. Its elements are  $C^\infty$ -diffeomorphisms  $f$  such that there exists a  $C^\infty$ -isotopy

$$(1.2) \quad t \in [0, 1] \mapsto f_t \in \text{Diff}(M) \quad \text{with } f_0 = \text{Id} \text{ and } f_1 = f.$$

Note that the discrete group  $\text{Diff}(M)/\text{Diff}^0(M)$  is the well known mapping class group. Define now

$$(1.3) \quad \text{Aut}^1(X) := \text{Aut}(X) \cap \text{Diff}^0(M).$$

There are obvious inclusions of groups

$$(1.4) \quad \text{Aut}^0(X) \subseteq \text{Aut}^1(X) \subseteq \text{Aut}(X)$$

In many examples, the first two groups are the same but differ from the third one (think of an elliptic curve). The purpose of this note is to describe an explicit family<sup>1</sup> of compact 3-folds  $\mathcal{X}_{a,b}$  such that

$$(1.5) \quad \text{Aut}^0(\mathcal{X}_{a,b}) \subsetneq \text{Aut}^1(\mathcal{X}_{a,b}) = \text{Aut}(\mathcal{X}_{a,b})$$

and

$$(1.6) \quad \text{Aut}^1(\mathcal{X}_{a,b})/\text{Aut}^0(X) = \mathbb{Z}_a$$

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<sup>1</sup>It is indexed by two integers satisfying  $b \geq 3a$  and  $a > 2$ .

Hence  $\text{Aut}^1(\mathcal{X}_{a,b})$  has  $a$  connected components, and this number can be chosen arbitrarily large.

Our main motivation comes from understanding the Teichmüller stack of  $M$ , that is the quotient stack of the set of complex operators on  $M$  modulo the action of  $\text{Diff}^0(M)$ , see [6] which contains a thorough study of this stack for dimension of  $M$  strictly greater than two. The isotropy group of the Teichmüller stack at some  $X$  is  $\text{Aut}^1(X)$ . So our result says that these isotropy groups may be not connected which is a source of complexity for the stack. Indeed, it appears as an obstacle for the Teichmüller stack and the classical Kuranishi space to be locally isomorphic, see [2].

The construction of the manifolds as well as the computation of their automorphism groups are elementary. As often when looking at explicit examples, the crux of the matter was to find the idea that makes everything work. We asked several specialists but they did not know any such example. We tried several classical examples but it always failed. Finally, we came across the good family when looking for deformations of Hopf surfaces over the projective line  $\mathbb{P}^1$  in connection with a different problem. The manifolds  $\mathcal{X}_{a,b}$  are such deformations with the following additional property. All the fibers are biholomorphic except for those that lie above 0 and above an  $a$ -th root of unity. Every automorphism must preserve these special fibers so must project onto  $\mathbb{P}^1$  as a rotation of angle  $2\pi k/a$  for some  $k$ . This explains the  $a$  connected components of the automorphism group. Finally, diffeomorphically there is no special fiber since  $\mathcal{X}_{a,b}$  is just a bundle. It is then not difficult to check that all rotations are allowed for diffeomorphisms so every automorphism is in  $\text{Diff}^0(M)$ .

## 2. THE MANIFOLDS $\mathcal{X}_{a,b}$ .

Let  $a$  and  $b$  be two nonnegative integers. Let  $\lambda$  be a non-zero complex number of modulus strictly less than one. For further use, we define the following two surfaces. Let  $X_0$  be the Hopf surface defined as  $\mathbb{C}^2 \setminus \{(0,0)\}$  divided by the group generated by the contraction  $(z,w) \mapsto (\lambda z, \lambda w)$ . Let  $X_1$  be the Hopf surface defined as  $\mathbb{C}^2 \setminus \{(0,0)\}$  divided by the group generated by the contraction  $(z,w) \mapsto (\lambda z + w, \lambda w)$ . These two Hopf surfaces are not biholomorphic, cf. [4].

We consider the vector bundle  $\mathcal{O}(b) \oplus \mathcal{O}(a) \rightarrow \mathbb{P}^1$ . Throughout the article, we make use of the charts

$$(2.1) \quad (t, z_0, w_0) \in \mathbb{C}^3 \quad \text{and} \quad (s, z_1, w_1) \in \mathbb{C}^3$$

subject to the relations

$$(2.2) \quad st = 1, \quad z_1 = s^b z_0, \quad w_1 = s^a w_0.$$

Let  $c \geq 0$  and let  $\sigma$  be a holomorphic section of  $\mathcal{O}(c)$ . In accordance with (2.1) and (2.2), we represent it in local charts by two holomorphic maps  $\sigma_0$  and  $\sigma_1$  satisfying  $\sigma_1(s) = s^c \sigma_0(t)$ . Let  $W$  be  $\mathcal{O}(b) \oplus \mathcal{O}(a)$  minus the zero section.

**Lemma 2.1.** *Assume that  $a - b - c \geq 0$ . Then, the holomorphic maps*

$$(2.3) \quad (t, z_0, w_0) \mapsto g_0(t, z_0, w_0) = (t, \lambda z_0 + \sigma_0(t)w_0, \lambda w_0)$$

and

$$(2.4) \quad (s, z_1, w_1) \mapsto g_1(s, z_1, w_1) = (s, \lambda z_1 + s^{b-a-c} \sigma_1(s) w_1, \lambda w_1)$$

defines a biholomorphism  $g$  of  $W$ .

*Proof.* Just compute in the other chart

$$\begin{aligned} g_1(s, z_1, w_1) &= (s, \lambda z_1 + s^{b-a-c} \sigma_1(s) w_1, \lambda w_1) \\ &= (1/t, s^b (\lambda z_0 + \sigma_0(t) w_0), s^a (\lambda w_0)) \end{aligned}$$

so  $g_0$  and  $g_1$  glue in accordance with (2.2).  $\square$

Consider now the group  $G = \langle g \rangle$ . It acts freely and properly on  $W$  and fixes each fiber of  $W \rightarrow \mathbb{P}^1$ . The quotient space  $W/G$  is thus a complex manifold. More precisely

**Proposition 2.2.** *The manifold  $W/G$  is a deformation of Hopf surfaces over  $\mathbb{P}^1$ . Moreover the fiber over  $t \in \mathbb{P}^1$  is biholomorphic to  $X_0$  if  $t$  is a zero of  $\sigma$ , otherwise it is biholomorphic to  $X_1$ .*

In particular,  $W/G$  is compact.

*Proof.* We already observed that the bundle map  $W \rightarrow \mathbb{P}^1$  descends as a holomorphic map  $\pi : W/G \rightarrow \mathbb{P}^1$ . It is obviously a proper holomorphic submersion, hence it defines  $W/G$  as a deformation of complex manifolds parametrized by the projective line. The fiber over  $t$  is  $\mathbb{C}^2 \setminus \{(0, 0)\}$  divided by the contracting map  $(z, w) \mapsto (\lambda z + \sigma(t)w, \lambda w)$ . If  $t$  is a zero of  $\sigma$ , then this is exactly the Hopf surface  $X_0$ . Otherwise, it is biholomorphic to  $X_1$ , see [4].  $\square$

**Definition 2.3.** Assume that  $c = 2a$  and that  $b \geq 3a$ . We denote by  $\mathcal{X}_{a,b}$  the manifold  $W/G$  corresponding to the choice

$$(2.5) \quad \sigma_0(t) = t^a \prod_{k=0}^{a-1} (t - \exp(2i\pi k/a)).$$

for  $t \in \mathbb{C}$ .

For the rest of the paper, we assume that  $a$  is strictly greater than 2. Observe that the condition  $b \geq 3a$  is nothing but  $b - a - c \geq 0$ .

### 3. COMPUTATION OF THE AUTOMORPHISM GROUPS.

We are in position to state and prove our main result.

**Theorem 3.1.** *The manifold  $\mathcal{X}_{a,b}$  satisfies*

$$(3.1) \quad \text{Aut}^0(\mathcal{X}_{a,b}) \simeq \left\{ \begin{pmatrix} \alpha & P \\ 0 & \alpha \end{pmatrix} \mid \alpha \in \mathbb{C}^*, P \in \mathbb{C}_{b-a}[X] \right\} / G$$

and

$$(3.2) \quad \begin{aligned} \text{Aut}(\mathcal{X}_{a,b}) &= \text{Aut}^1(\mathcal{X}_{a,b}) \\ &\simeq \mathbb{G}_a \times \left\{ \begin{pmatrix} \alpha & P \\ 0 & \alpha \end{pmatrix} \mid \alpha \in \mathbb{C}^*, P \in \mathbb{C}_{b-a}[X] \right\} / G \end{aligned}$$

where  $\mathbb{G}_a$  is the group of  $a$ -th roots of unity,  $\mathbb{C}_{b-a}[X]$  is the space of complex polynomials with degree at most  $b-a$  and where the product in (3.2) is given by

$$(3.3) \quad \left( r, \begin{pmatrix} \alpha & P \\ 0 & \alpha \end{pmatrix} \right) \cdot \left( r', \begin{pmatrix} \beta & Q \\ 0 & \beta \end{pmatrix} \right) = \left( rr', \begin{pmatrix} \alpha\beta & \alpha Q \circ r + \beta P \circ r \\ 0 & \alpha\beta \end{pmatrix} \right)$$

We note the immediate corollary

**Corollary 3.2.** *The group  $\text{Aut}^1(\mathcal{X}_{a,b})$  has  $a \geq 3$  connected components and the quotient  $\text{Aut}^1(\mathcal{X}_{a,b})/\text{Aut}^0(\mathcal{X}_{a,b})$  is isomorphic to the cyclic group  $\mathbb{Z}_a$ .*

Theorem 3.1 will be proved through a succession of Lemmas.

**Lemma 3.3.** *Let  $f$  be an automorphism of  $\mathcal{X}_{a,b}$ . Then it respects  $\pi$  and descends as an automorphism  $h$  of  $\mathbb{P}^1$ .*

*Proof.* Choose a fiber of  $\mathcal{X}_{a,b} \rightarrow \mathbb{P}^1$  isomorphic to  $X_1$ . Restrict  $f$  to it and compose with the projection onto the projective line. This gives a holomorphic map from  $X_1$  to  $\mathbb{P}^1$ , hence a meromorphic function on  $X_1$ . But the algebraic dimension of  $X_1$  is zero, see [3], so this map is constant. In other words,  $f$  sends the  $\pi$ -fibers isomorphic to  $X_1$  onto the  $\pi$ -fibers. By density of these fibers,  $f$  sends every  $\pi$ -fiber onto a  $\pi$ -fiber so descends as an automorphism  $h$  of  $\mathbb{P}^1$ .  $\square$

**Lemma 3.4.** *The automorphism  $h$  is a power of the rotation at 0 of angle  $2\pi/a$ .*

*Proof.* Note that  $f$  must send a  $\pi$ -fiber biholomorphic to  $X_0$  onto a  $\pi$ -fiber biholomorphic to  $X_0$ . Now the set of such fibers is the set of  $a$ -th roots of unity plus zero by Proposition 2.2 and (2.5). It follows from Lemma 3.3 that the automorphism  $h$  is an automorphism of the projective line which preserves this set. Hence it is a power of the rotation at 0 of angle  $2\pi/a$ .  $\square$

Lift  $f$  as an automorphism  $F$  of the universal covering  $W$  of  $\mathcal{X}_{a,b}$ . We denote by  $(F_0, F_1)$  its expression in the charts (2.1).

**Lemma 3.5.** *In the charts (2.1), the lifting  $F$  has the following form*

$$(3.4) \quad F_0(t, z_0, w_0) = (r^k t, \alpha z_0 + \tau_0(t)w_0, \alpha w_0)$$

and

$$(3.5) \quad F_1(s, z_1, w_1) = (r^{-k} s, r^{-kb}(\alpha z_1 + \tau_1(s)w_1), \alpha w_1)$$

where  $r = \exp(2i\pi/a)$ ,  $k$  is an integer,  $\alpha$  a complex number and  $\tau = (\tau_0, \tau_1)$  is a section of  $\mathcal{O}(b-a)$ .

*Proof.* The first coordinate in (3.4) comes from Lemma 3.4. For the two other coordinates, recall from [7] that the automorphism group of  $X_0$  is  $\text{GL}_2(\mathbb{C})$  (modulo quotient by the group generated by the contraction) and that of  $X_1$  is the group of upper triangular matrices with both entries on the diagonal equal (modulo quotient by the group generated by the contraction).

Hence the general form of  $F_0$  is

$$(3.6) \quad F_0(t, z_0, w_0) = (r^k t, \alpha_0(t)z_0 + \tau_0(t)w_0, \alpha_0(t)w_0)$$

for  $\alpha_0$  and  $\tau$  two holomorphic functions. But in the other chart, using the same more general form of (3.5), we must have

$$\begin{aligned} F_1(s, z_1, w_1) &= (r^{-k}s, r^{-kb}(\alpha_1(s)z_1 + \tau_1(s)w_1), \alpha_1(s)w_1) \\ &= (1/(r^k t), (r^{-k}s)^b(z_0\alpha_1(1/t) + \tau_1(1/t)s^{a-b}w_0), s^a\alpha_1(1/t)w_0) \end{aligned}$$

which extends at  $s = 0$  and glues with (3.6) if and only if  $\alpha = (\alpha_0, \alpha_1)$  is a constant and  $\tau = (\tau_0, \tau_1)$  is a section of  $\mathcal{O}(b-a)$ .

It remains to check whether these automorphisms really descend as automorphisms of  $\mathcal{X}_{a,b}$ , i.e. whether they commute with the contraction  $g$  of Lemma 2.1. We compute

$$\begin{aligned} g_0 \circ f_0(t, z_0, w_0) &= (r^k t, \lambda \alpha z_0 + \lambda \tau_0(t)w_0 + \alpha \sigma_0(r^k t)w_0, \lambda \alpha w_0) \\ &= f_0 \circ g_0(t, z_0, w_0) \end{aligned}$$

since  $\sigma$  is  $\mathbb{G}_a$ -invariant, cf. (2.5). A similar computation holds in the  $(s, z_1, w_1)$ -coordinates, so finally all these automorphisms descend.  $\square$

**Lemma 3.6.** *An automorphism of  $\mathcal{X}_{a,b}$  is in the connected component of the identity if and only if it descends as the identity of  $\mathbb{P}^1$ .*

*Proof.* If an automorphism  $f$  of  $\mathcal{X}_{a,b}$  is in the connected component of the identity, then by Lemma 3.4 its projection  $h$  is isotopic to the identity through rotations of angle  $2i\pi k/a$ . This is only possible if  $h$  is the identity. Conversely, if  $h$  is the identity, it is easy to see that in Lemma 3.5 we can move  $\alpha$  to 1 and  $\tau$  to the zero section and obtain a path of automorphisms from  $f$  to the identity.  $\square$

**Lemma 3.7.** *Every automorphism of  $\mathcal{X}_{a,b}$  is isotopic to the identity through  $C^\infty$ -diffeomorphisms.*

*Proof.* Let  $B_{a,b}$  be the bundle over  $\mathbb{P}^1$  with fiber  $X_0$  obtained by taking the quotient of  $W$  by the group generated by the  $\lambda$ -homothety in the fibers. Observe that  $\mathcal{X}_{a,b}$  can be deformed to  $B_{a,b}$  putting a parameter  $\epsilon \in \mathbb{C}$  and considering the family

$$(3.7) \quad W \times \mathbb{C} / \langle \tilde{g} \rangle \longrightarrow \mathbb{C}$$

where the action is given by (we just write it down in the first chart):

$$(3.8) \quad (t, z_0, w_0, \epsilon) \mapsto \tilde{g}_0(t, z_0, w_0, \epsilon) = (t, \lambda z_0 + \epsilon \sigma_0(t)w_0, \lambda w_0, \epsilon)$$

Hence  $\mathcal{X}_{a,b}$  is  $C^\infty$ -diffeomorphic to  $B_{a,b}$ . More precisely, let  $\mathcal{X}_{a,b}^\epsilon$  be the fiber of the family (3.7) over  $\epsilon$ . Then  $\mathcal{X}_{a,b}^0 = B_{a,b}$ , and  $\mathcal{X}_{a,b}^1 = \mathcal{X}_{a,b}$  and by Ehresmann's Lemma there is an isotopy of  $C^\infty$ -diffeomorphisms  $\phi_\epsilon$  from  $\mathcal{X}_{a,b}^\epsilon$  onto  $\mathcal{X}_{a,b}^1$  with  $\phi_1$  equal to the identity.

Let  $f$  be an automorphism of  $\mathcal{X}_{a,b}$ . It is easy to check, using Lemma 3.5, that  $f$  is still an automorphism of  $B_{a,b}$  and of all the  $\mathcal{X}_{a,b}^\epsilon$ . We are saying that the map

$$(3.9) \quad F(t, z_0, w_0, \epsilon) = (F_\epsilon(t, z_0, w_0), \epsilon) = (f(t, z_0, w_0), \epsilon)$$

is an automorphism of the whole family which induces  $f$  on each fiber. The isotopy  $\phi_\epsilon \circ F_\epsilon \circ (\phi_\epsilon)^{-1}$  joins the automorphism  $f = F_1$  of  $\mathcal{X}_{a,b}$  and  $\phi_0 \circ F_0 \circ (\phi_0)^{-1}$  through  $C^\infty$ -diffeomorphisms.

But now, at  $\epsilon = 0$ , since  $B_{a,b}$  is a holomorphic bundle, we may take any rotation at 0 as map  $h$  and thus construct a path of automorphisms  $g_t$  of  $B_{a,b}$  between  $F_0$  and the identity. Combining the isotopy  $\phi_0 \circ g_t \circ (\phi_0)^{-1}$  with the previous one, we obtain that the automorphism  $f$  of  $\mathcal{X}_{a,b}$  is isotopic to the identity through  $C^\infty$ -diffeomorphisms.  $\square$

The proofs of Theorem 3.1 and Corollary 3.2 follow easily from the previous lemmas.

#### 4. FINAL COMMENTS.

There are several open questions left on this topic. First the manifolds  $\mathcal{X}_{a,b}$  are not even Kähler and it would be interesting to have a similar example with projective manifolds. Also, it would be interesting to have examples with  $\text{Aut}^0(X)$  reduced to zero, but  $\text{Aut}^1(X)$  not, cf. [2] and the subsequent literature. Finally, the most exciting would be to find an example with  $\text{Aut}^1(X)/\text{Aut}^0(X)$  infinite. As pointed out to us by S. Cantat, this cannot happen for Kähler manifolds, since the kernel of the action of  $\text{Aut}(X)$  on the cohomology contains  $\text{Aut}^0(X)$  as a subgroup of finite index [5]. But in the non-Kähler world everything is possible.

For all these additional questions (except perhaps for the first one), it seems that a really different type of examples is needed.

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